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Three Logics of Subjunctive Conditionals*

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In this paper I wish to do two things. First, I wish to discuss three logics which differ from standard logics by including "It is false that (if p then p is false)" as a theorem. I shall call this theorem the Law of Conditional Non-Contradiction. Secondly, I wish to defend the thesis that logics of this sort serve to axiomatize contrary-to-fact and subjunctive conditionals.

§ 1. The content and historical precedents for the Law of Conditional Non-Contradiction. Intuitively the Law of Conditional Non-Contradiction appears a reasonable theorem, if we consider what is being denied. What is denied is the statement "If p were the case, then p would not be the case.", and this conditional statement form appears as self-contradictory as the conjunctive statement "p is the case and p is not the case." The denial of this schema, then, makes as plausible *prima facie* candidate for universal logical truth as the denial of "both p and not p". Although mathematicians may have some fears that the universal rejection of this principle might inhibit the development of *Reductio ad Absurdum* proofs, such fears have no foundation.

Further, the Law of Conditional Non-Contradiction is not new to formal logic. Aristotle, in Book II, Chapter 4 of the *Prior Analytics*, rejected an argument on the grounds that the premises led to the conclusion "if B is not great, then B is great", which conclusion, he said, was impossible. Maier commented on this passage, saying that "if not p then p" is a connexion contrary to the Law of Contradiction and therefore absurd.[1] Lukasiewicz, rejected

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[1]Maier, H., *Die Syllogistic des Aristotsles*, Vol. ii a, Tubingen, 1900, p. 331.

Maier's comment, because in modern logic the Law of Contradiction denies the

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conjunction "p and not p" but not the conditional "if p then not p". Nevertheless, Lukasiewicz held that Aristotle, though in error, used this Law of Conditional Non-Contradiction as one of three implicit laws of propositional logic in his proofs of imperfect syllogisms. (The other two laws were a Law of transposition and the Law of the Hypothetical Syllogism.)[2] That such a

[2]Lukasiewicz, Jan, Aristotle's Logic, p. 49-50.

principle is indeed implicit in Aristotle's theory of the syllogism is given further support by the fact that the traditional square of opposition and the traditional "weak" syllogisms need not be altered with such a conditional, though they are incompatible with the introduction of the truth-functional conditional.[3]

[3]See McCall, Storrs, "Connexive Implication and the Syllogism" and Angell, R. B., "Material Conditionals and Modern vs Traditional Syllogistic", paper presented at the Association for Symbolic Logic, New York City, December 28, 1965. Abstract forthcoming in JSL.

Kneale also attributed this principle to Aristotle,[4] though he, like Lukasiewicz

[4]Kneale, W., "Aristotle and the Consequentia Mirabilis", Journal of Hellenic Studies, 77, part i, 1957, pp. 62-66.

rejected it. McCall, who accepts the principle in his "connexive" logic, is sufficiently confident of its Aristotelian roots to give it the name Aristotle.[5]

[5]McCall, Storrs, Doctoral Dissertation, Oxford University; also "Connexive Logic".

Since Principia Mathematica, this principle has been presented, independently apparently of Aristotle, by Nelson, who, in seeking a notion of entailment free of the so-called paradoxes of both material and strict implication, proposed that "it is false that p entails not p" should be a theorem of intensional logic.[6] Although his notion of entailment was defined in terms of the undefined

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[6]Nelson, E. J., "Intensional Relations", Mind, n.s., Vol. 39, (1930) pp. 449.

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relation "p is inconsistent with q" where p and q took propositions as their values, he has held that his law may be interpreted as denying a conditional, i.e., as the Law of Conditional Non-Contradiction. Nelson did not, however, establish formally the consistency of logics containing this principle.

Lukasiewicz was absolutely correct in noting that this Law of Conditional Non-Contradiction does not hold for the truth-functional conditional of the classical logic of Frege and Russell. In classical logic the counter-instance, '$(p.-p) \supset -(p.-p)$', is a theorem, and any logic which contains as a theorem '$-(p \rightarrow -p)$' must therefore be independent of classical logic. But this theorem cannot be dismissed as an absolute error. Apart from philosophical prejudices, the possibility and utility of a logic which contains such a principle is a fair subject for investigation. Logics containing this law can be constructed which are not only consistent, and independent of classical logic, but are complete with respect to standard logic, containing all of the laws of classical sentential calculus. We shall examine three such systems.

§ 2. Angell's system PA1. In 1962 the author published a paper entitled "A Propositional Logic of Subjunctive Conditionals"[7]. The chief

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[7]JSL, Vol. 27, September 3, 1962.

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contribution of this paper was to demonstrate the consistency of a propositional logic which contains both the Law of Conditional Non-Contradiction, the entire classical sentential calculus (construed as a logic of conjunction and denial) and most of the principles of traditional logic involving conditionals.

The system included conjunction and denial, plus an arrow for the type of conditional under consideration, as primitive signs, classical abbreviations for definitions, and ten axioms. The last axiom was called the Law of Subjunctive Contrariety, A10.

$$((p \rightarrow q) \rightarrow -(p \rightarrow -q))$$

From it the Law of Conditional Non-Contradiction follows by putting '$(p \rightarrow p)$' in the antecedent. For transformation rules PA1 had Modus Ponens and a Rule of Adjunction.[8] This system was shown to be consistent by utilizing the following

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[8] The following is the logical base of PA1.

I. <u>Primitive Symbols</u>
   1) Grouping devices: ( )
   2) Constants: $-$, $.$, $\rightarrow$
   3) Variables: p, q, r, s, p_1, q_1, r_1, ...

II. <u>Rules of Formation</u>
   F_1. A single variable by itself is a wff.
   F_2. If any formula S is a wff, then $-S$ is a wff.
   F_3. If S and S' are wffs, then $(S.S')$ is a wff.
   F_4. If S and S' are wffs, then $(S \rightarrow S')$ is a wff.

III. <u>Abbreviations</u>
   D_1. $(S \vee S') = \text{df} -(-S.-S')$
   D_2. $(S \supset S') = \text{df} -(S.-S')$
   D_3. $(S \equiv S') = \text{df} ((S \supset S').(S' \supset S))$

IV. <u>Primitive Formulas</u>
   A_1. $((q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$
   A_2. $((p \rightarrow q) \rightarrow ((r.p) \rightarrow (q.r)))$
   A_3. $((p \rightarrow -(q.r)) \rightarrow ((q.p) \rightarrow -r))$
   A_4. $((p.(q.r)) \rightarrow (q.(p.r)))$
   A_5. $((p \rightarrow -q) \rightarrow (q \rightarrow -p))$
   A_6. $(--p \rightarrow p)$
   A_7. $((p \rightarrow q) \rightarrow -(p.-q))$
   A_8. $-((p.q).-p)$
   A_9. $-(p.-(p.p))$
   A_{10}. $((p \rightarrow q) \rightarrow -(p \rightarrow -q))$

V. <u>Rules of Transformation</u>
   R_1. If $\vdash S$ and $\vdash (S \rightarrow S')$, then $\vdash S'$.
   R_2. If $\vdash S$ and $\vdash S'$, then $\vdash (S.S')$.
   R_3. If $\vdash S$ and if v is a propositional variable occurring in S, then if S' is got by replacing all occurrences of v in S by any wff, T, then $\vdash S'$.
   R_4. If $\vdash S$ and S' is got by replacing any part, or all, of S by an expression equivalent through rules of abbreviation, then $\vdash S'$.

matrices with 0 and 1 as designated values.

-	p
3	0
2	1
1	2
0	3

(p.q)	0	1	2	3
0	1	0	3	2
1	0	1	2	3
2	3	2	3	2
3	2	3	2	3

(p→q)	0	1	2	3
0	1	2	3	2
1	2	1	2	3
2	1	2	1	2
3	2	1	2	1

The advantages and disadvantages of PA1 as a formalization of propositional logic may be set forth under four headings. In its favor were four considerations. First, by containing the classical sentential calculus, it became immune to any charge that it would lead to a logic less powerful than classical logic. Whatever deduction procedures or logical truths belonged to classical logic, belong with exactly the same sense to PA1 as well.

Secondly, the new conditional was capable of expressing many, though not all, of the commonplace principles of traditional logic, e.g., mixed and pure hypothetical and alternative and disjunctive syllogisms, Complex Dilemmas, Sorites, the laws of double negation and transposition, and permutation, association and commutation for conjunctions and disjunctions.

Thirdly, the new conditional was provably free of what has been called paradoxes of material and strict implication. Inspection of the conditional matrix shows that no proposition, contradictory, tautologous, or otherwise, can imply, or be implied by, every proposition. All previously listed "paradoxes" of this sort failed for the new conditional, and were eliminated from classical logic by withdrawal of the rule interpreting the '⊃' as meaning "if...then...".

Fourthly, PA1 added certain classes of rather plausible theorems which cannot be derived in classical logic. Consider the class of logically true conditional statements; it would seem that if, for example, "if (p and q) then (q and p)" is a logical truth, then "If (p and q) then it is false that (q and p)" should be logically false, especially when expressed in the subjunctive mood. In any case, every logically true subjunctive conditional in PA1 implies another theorem in PA1 which consists of the denial of the first conditional with its consequent denied. Though classical logic leads us to reject this inference where the conditional is truth-functional, many writers have held that in contrary-to-fact and subjunctive, or other "ordinary" conditionals, such an inference is considered sound.

But as we shall see, after considering McCall's related system, CC1, both PA1 and CC1, suffer from several defects which render them unacceptable.

§ 3. McCall's System CC1. The chief disadvantages of PA1 had to do with its incompleteness in several senses. Though complete with respect to classical logic, it was not shown Post-complete, or functionally complete. McCall produced an axiom set, CC1, which he called a "connexive logic", which was equivalent to PA1 with five axioms added so as to prove that all tautologies in the matrices assigned to PA1 were provable.[9] From this, McCall proved his system Post-

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[9]McCall's Axiom set in his pre-publication draft of "Connexive Implication", 1965, was (he used Polish notation):

1. $((p\rightarrow q)\rightarrow((q\rightarrow r)\rightarrow(p\rightarrow r)))$
2. $(((p\rightarrow p)\rightarrow q)\rightarrow q)$
3. $((p\rightarrow q)\rightarrow((p.r)\rightarrow(r.q)))$
4. $((q.q)\rightarrow(p\rightarrow p))$
5. $((p.(q.r))\rightarrow(q.(p.r)))$
6. $((p.(p\rightarrow p))\rightarrow p)$
7. $((p.p)\rightarrow((p\rightarrow p)\rightarrow(p.p)))$
8. $(p\rightarrow((p.p).p))$
9. $((-q\rightarrow -p)\rightarrow(p\rightarrow q))$
10. $((p\rightarrow --p))$
11. $((p.-(p.-q))\rightarrow q)$
12. $(-p\vee(p.p))$
13. $((-p\vee((p\rightarrow p)\rightarrow p)\vee(((p\rightarrow p)\vee(p\rightarrow p))\rightarrow p))$
14. $((p\rightarrow p)\rightarrow -(p\rightarrow -p))$

This was equivalent to PA1 with the following five axioms, all of which are contained in the classical implication fragment, added:

11. $(p\rightarrow((p.p).p))$
12. $((p.p)\rightarrow((p\rightarrow p)\rightarrow(p.p)))$
13. $(((p\rightarrow p)\rightarrow q)\rightarrow q)$
14. $((q.q)\rightarrow(p\rightarrow p))$
15. $(-p\vee(((p\rightarrow p)\rightarrow p)\vee((q\vee q)\rightarrow p)))$

complete, though not functionally complete. Also, in his Doctoral Dissertation, McCall showed that any system containing the Law of Conditional Non-Contradiction (which he called Aristotle), was independent of the implication-negation fragments not only of classical logic, but of Lewis's Systems S3-S5, of Heyting's Intuitionistic calculus, of Church's Weak Positive Implicational Calculus with strong negation, of Lukasiewicz's three-valued logic, of Ackermann's 'rigorous' implication, and Anderson and Belnap's system of entailment.[10]

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[10] The proof consisted in showing that all of these systems contained in their implication-negation fragment

1. $((p\to q)\to((q\to r)\to(p\to r)))$
2. $(((p\to p)\to q)\to q)$
3. $((p\to -q)\to(q\to -p))$
4. $((-p\to p)\to p)$
5. $((p\to q)\to(((p\to q)\to r)\to((r\to s)\to s)))$

from which, letting 'P' represent '$(p\to p)$', we get a counter instance of the Law of Conditional Non-Contradiction:

6.	$((((-P\to P)\to P)\to -P)\to(P\to -((-P\to P)\to P)))$	[From 3]
7.	$((((-P\to P)\to P)\to -P)\to -((-P\to P)\to P))$	[1-(6→(2→7))]
8.	$(((-P\to P)\to P)\to -(((-P\to P)\to P)\to -P))$	[3-(7→8)]
9.	$((((-P\to P)\to P)\to -P)\to((-P\to P)\to P))$	[5-(4→9)]
10.	$((((-P\to P)\to P)\to -P)\to -(((-P\to P)\to P)\to -P))$	[1-(9→(8→10))]

McCall's system CCl added some desirable theorems, e.g., $(p\to((p.p).p))$. But it also contained some clearly undesirable theorems, e.g., the invalid syllogism (established by truth-tables)

$$((p\to q).(r\to q))\to -(p\to -r))$$

which would validate the argument:

If <u>a</u> is a dog then <u>a</u> is an animal
If <u>a</u> is a cat then <u>a</u> is an animal
Hence, It is false, that if <u>a</u> is a dog, then <u>a</u> is not a cat.

which PA1, had not therefore provably excluded.

§ 4. <u>Defects of PA1 and CC1</u>. Aside from the defect just mentioned, CC1 and PA1 shared two sorts of defects. One sort was eliminable by choice of a different matrix for conjunction. The other sort, however, stemmed from problems implicit in the type of system sought.

The first sort of defect due to the conjunction truth-table, included the following: (1) both systems excluded the principles

$$(1)\ ((p.p) \rightarrow p) \quad \text{and} \quad (p \rightarrow (p.p)).$$

This was anamolous because CC1 included

$$(2)\ (((p.p).p) \rightarrow p) \quad \text{and} \quad (p. \rightarrow (p.(p.p))$$

so that any proposition p was equivalent to any conjunction consisting of an odd number of occurrences of it, but neither implied nor was implied by any conjunction consisting of an even number of occurrences of p. Also it excluded all distributive laws for conjunction and alternation

$$((p \vee (q.r)) \rightarrow ((p \vee q).(p \vee r)))$$
$$((p.q) \vee (p.r)) \rightarrow (p.(q \vee r)))$$

There seemed to be no philosophical justification for such exclusions and irregularities.

These problems were removable by replacing the conjunction matrix by another sort, e.g.

(p.q)	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

which was Lewis's, or

(p.q)	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

But such matrices did not satisfy Axioms 2, 3, 7, (or 12, 14 and 15) of PA1 ~~(or CC1)~~.

The second, more crucial problem, however, concerns incompatibilities which arise when the Law of Conditional Non-Contradiction is taken in conjunction with certain other desired principles and laws.

If we assume that any plausible logic must contain normal rules of substitution and, at least,

(1) a rule of the hypothetical syllogism, i.e., "If '$(p \rightarrow q)$' is true and '$(q \rightarrow r)$' is true, then '$(p \rightarrow r)$' is true";

(2) A rule of transposition, i.e., "If '$(p \rightarrow q)$' is true, then '$(-q \rightarrow -p)$' is true";

and (3) the Commutative Law for Conjunction "$((p.q) \rightarrow (q.p))$"

and then we add the Law of Conditional Non-Contradiction,

(4) $-(p \rightarrow -p)$

it is easily shown that we cannot consistently include the Law of Simplification:

$((p.q) \rightarrow q)$ (hypothesis)

For, by substitutions of '$-p$' for 'q', we get the following proof:

(6) $(p.-p) \rightarrow (-p.p)$	[by (3), subst.]
(7) $(-p.p) \rightarrow p$	[by hyp. sub.]
(8) $(p.-p) \rightarrow p$	[(6), (7), Syll]
(9) $-p \rightarrow -(p.-p)$	[(8), Trans.]
(10) $(p.-p) \rightarrow -p$	[hyp., sub.]
(11) $(p.-p) \rightarrow -(p.-p)$	[(10), (9), Syll]

and the conclusion is the contradictory of the substitution instance of (3),

(12) $-((p.-p).-(p.-p))$

Thus if we are to defend a system of logic which contains the Law of Conditional Non-Contradiction, we must be prepared to give some credence to that thesis that '$(p.q)$' does not always imply or entail q. This is the first inescapable problem which must be resolved.

A second inescapable difficulty arises in connection with the law, as distinct from the rule, of affirming the disjunct. In some sense this rule is indispensable if standard logic is to be included in our system, for, by classical definitions and double-negation principles, it is clear that the following rules are equivalent:

"If p and $(p \supset q)$ are true, then q is true."

"If $-p$ and $(p \vee q)$ are true, then q is true."

"If p and $-(p.q)$ are true, then $-q$ is true."

The last rule is the more basic since it, unlike the others, does not assume '$(--p \rightarrow p)$' which intuitionists reject. That this rule of affirming the disjunct should be admitted, must be granted. But the admission of the corresponding law of logic

Hypothesis: $((p.-(p.q)) \rightarrow -q)$ (Law of Affirming the Disjunct)

leads to difficulties. Let us suppose we add to the essential elements in our logic a rule of the Factor:

(R3) "If $(p \leftrightarrow q)$ is true, then '$((p.r) \rightarrow (q.r))$' is true"

This rule is essential if we are to establish a rule for the substitution of p for q when $(p \leftrightarrow q)$ is a theorem.

Further, we shall assume that no objection will be raised to the Law of Double Negation, which is admitted by both classical ~~and intuitionist~~ logic:

(A3) $(p \leftrightarrow --p)$

and that the following laws of conjunction are both beyond exception,

(A4 $((p.p) \rightarrow p)$

(A5) $(p \rightarrow (p.p))$

so that equivalences can be set up, including the following:

(1) $(-p \leftrightarrow -(p.p))$

And now the impossibility of adding the third version of the Law of Denying the Disjunct with these assumptions, can be proven:

(2)	$((p.-p) \rightarrow (p.-(p.p)))$	(by (1) and Rule of Factor)
(3)	$((p.-(p.p)) \rightarrow -p)$	Hyp - Law of Affirming the Disjunct
(4)	$((p.-p) \rightarrow -p)$	((2), (3), Rule of Syll)
(5)	$((-p.--p) \rightarrow --p)$	((4), Substitution)
(6)	$((-p.p) \rightarrow (-p.--p))$	(Double Neg., and Rule of Factor) [error – used wrong rule of factor]
(7)	$((p.-p) \rightarrow (-p.p))$	(Law of Commutation)
(8)	$((p.-p) \rightarrow --p)$	((7), (6), (5), Syllogism, twice)
(9)	$(--p \rightarrow -(p.-p))$	((4), Rule of Transposition)
(10)	$((p.-p) \rightarrow -(p.-p))$	((8), (9), Rule of Syllogism)

The conclusion, (10), is again the contradictory of an instance of the Law of Conditional Non-Contradiction.

The intuitive argument against admitting the <u>Law</u> of Affirming the Disjunct is closely related to the argument for rejecting the law '$(p.-p) \rightarrow -p$)'. For if it be granted the '$(p.p)$' is equivalent to 'p', then the law '$((p.-(p.p)) \rightarrow -p)$' must be accepted as equivalent to '$((p.-p) \rightarrow -p)$'. Thus, the <u>Law</u> of Affirming the Disjunct, which incidently is also missing in the systems of Ackermann, Anderson and Belnap (and the omission of which is defended by Anderson) must be rejected. But certainly there must be a rule, "If p and $-(p.q)$ are true, then $-q$ is true".

There are other questions that might be raised about PA1 and CC1 with regard to the exclusion of laws which have analogues in the classical calculus. But these two sorts of problems are sufficient, perhaps, to show that PA1 and CC1 are in need of revisions.

§ 5. The System PA2. We turn next to a new system, PA2, which attempts to solve some of the defects above. This system, like PA1, contains the Law of Conditional Non-Contradiction (Axiom 7), but it is weaker in its non-classical theorems than CC1 since it replaces the Law of Subjunctive Contrariety, Axiom 10

$$((p \rightarrow q) \rightarrow -(p \rightarrow -q))$$

with a theorem which asserts

$$-((p \rightarrow q).(p \rightarrow -q)).$$

This however, still permits inferences from any true '$p \rightarrow q$' to "it is false that $p \rightarrow -q$". PA2 differs more radically from PA1 and from Classical Logic, however, in including a truth sign, or truth-operator, which makes it possible to express in the object-language of PA2 such theorems as

(1) If (p and q) is true then p is true.

(2) If p is true and -(p and q) is true, then -q is true.

Although these English translations are identical with what are usually called metalogical rules of inference, in PA2 they are theorems, or laws. They make it possible to include many theorems related to the principle of simplification which found no place in PA1. Expressions of the form ⌜T(S)⌝ may be read rigorously as "It is true that S" without using quotes, just as ⌜-(S)⌝ may be read "It is false that S" without using quotes.

The following are the axioms of PA2:

I. Primitive Signs: T, -, ., $\rightarrow$

II. Rules of Formation:
F_1. Any variable is a wff.
F_2. If S is wf, then ⌜-S⌝ and ⌜TS⌝ are wffs.
F_3. If S and S' are wf, then ⌜(S.S')⌝ and ⌜(S$\rightarrow$S')⌝ are wffs.

III. Abbreviations:
D_1. ⌜(S V S')⌝ = df ⌜-(-S.-S')⌝
D_2. ⌜(S $\supset$ S')⌝ = df ⌜-(S.-S')⌝
D_3. ⌜(S $\equiv$ S')⌝ = df ⌜((S $\supset$ S').(S' $\supset$ S))⌝
D_4. ⌜(S $\leftrightarrow$ S')⌝ = df ⌜((S$\rightarrow$S').(S'$\rightarrow$S))⌝

IV. **Axioms**

A_1. $T((p\rightarrow q)\rightarrow((r\rightarrow p)\rightarrow(r\rightarrow q)))$
A_2. $T((p\leftrightarrow q)\rightarrow((r.p)\rightarrow(q.r)))$
A_3. $T((p.p)\leftrightarrow p)$
A_4. $T((p.(q.r))\rightarrow(q.(p.r)))$
A_5. $T((p\rightarrow -q)\rightarrow(q\rightarrow -p))$
A_6. $T(--p\rightarrow p)$
A_7. $T-(p\rightarrow -p)$
A_8. $T(T(p.q)\rightarrow Tp)$
A_9. $T(Tp\rightarrow(Tq\rightarrow T(p.q)))$
A_{10}. $T((Tp\rightarrow Tq)\leftrightarrow T-(p.-q))$
A_{11}. $T((p\rightarrow q)\rightarrow(Tp\rightarrow Tq))$
A_{12}. $T(Tp\rightarrow(T(p\rightarrow q)\rightarrow Tq))$

I see no need for any rules of inference other than substitution. A common way of expressing the rule of Modus Ponens is;

MP "If p is true then if (if p then q) is true, then q is true".

But this is expressed precisely in PA2 as Axiom 9.

Axiom #2 $(Tp\rightarrow(T(p\rightarrow q)\rightarrow Tq))$

which is a tautology. But this is not the same as

$((p.(p\rightarrow q))\rightarrow q)$

which is not a theorem. There is, to be sure, a difference between using a set of axioms, and simply having them there. Presumably, there is need to teach the reader how to use these axioms in order to get theorems. In this sense rules of deriving must be explained, and such explanations are not part of the axiom set. But neither are such explanations included in the classical statements of so-called rules of inference. The use of the T-operator makes the necessary distinction between conditionals which can operate as principles of inference in inferring and those that can not. '$(p\rightarrow q)$' and '$((p.q)\rightarrow p)$' for example, can not be used as a premiss to get a logically true conclusion; '$T(--p\rightarrow p)$' because an axiom, and because of the T in front, can be used in an inferential conditional like T9, since it has the over-all form of the antecedent 'T(S)'. This explains how these axioms are to be syntactically manipulated. The later proof of consistency is simply a proof showing that if these axioms are manipulated symbolically in the right ways, we will never get both 'S' and '-S' as results.

Several considerations may be made in defense of PA2.

First, it, like CC1, is provably consistent. Consistency is established by the following four-valued matrices, with the numbers 1 and 2 as designated values.

T	p
2	1
2	2
4	3
4	4

-	p
4	1
3	2
2	3
1	4

(p→q)	1	2	3	4
1	2	4	4	4
2	4	2	4	4
3	2	4	2	4
4	4	2	4	2

(p.q)	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

All of the axioms A1-A7 take only designated values and tautology is preserved through uniform substitution as well as through the use of any of the axioms 8, 9, 10 and 12 that are used to operate in inference.

Secondly, it, like CC1 and PA1, PA2 yields a derived rule providing for the substitutivity of equivalents, where S is equivalent to S' if and only if (S↔S') is a logical truth. This rule is handy in a variety of ways.

Thirdly, like CC1 and PA1, the system is complete with respect to classical logic. Thus all truths and laws of Principia Mathematica are asserted as truths in this system in precisely the same sense in which they are asserted in Principia Mathematica. Whether the words "if...then..." will be used for expressions of the form ⌜(p ⊃ q)⌝ is a matter of semantic convention. Provided a clear distinction is drawn between '⊃' as a "truth-functional 'if...then...'" and '→' as some other kind of "if...then..." (perhaps, as I propose, a "subjunctive 'if...then...'"), there is no harm in it. But if '⊃' is used as an "if...then..." it must be qualified as "truth-functional", and if it is not so qualified, it should not be associated with "if...then..." at all. The system here presented is richer than Principia; it cannot be reduced to it, and it contains theorems which Principia does not contain.

Fourthly, although the pure implicational fragment is not as rich as PA1 and CC1, through the use of the T-operator a completely satisfactory set of inferential implications are forthcoming. Thus although PA1 and CC1 contained

(1) $((p.(p\rightarrow q))\rightarrow q)$

(2) $(p.(-p\vee q))\rightarrow q)$

while failing to include

(1') $((p.(p\rightarrow q))\rightarrow(p.q))$

(1'') $(p\rightarrow((p\rightarrow q)\rightarrow q))$

(2') $((p.(-p\vee q)\rightarrow(p.q))$

(2'') $(p\rightarrow((-p\vee q)\rightarrow q))$

PA2 contains inferential forms for all of these, though it lacks (1) and (2) as they stand. Thus we have in PA2:

(1) $(T(p.(p\rightarrow q))\rightarrow Tq)$

(2) $(T(p.(-p\vee q))\rightarrow Tq)$

(3) $(T(p.(p\rightarrow q))\rightarrow T(p.q))$

(4) $(Tp\rightarrow(T(p\rightarrow q)\rightarrow Tq)$

(5) $(T(p.(-p\vee q))\rightarrow Tq)$

(6) $(Tp\rightarrow(T(-p\vee q)\rightarrow Tq))$

In fact, as can be seen by consideration of Axiom 10, Definition 2, the completeness of PA2 relative to classical logic, and the derived rule for substituting equivalents, wherever we have proved a logical implication of classical logic,(i.e., wherever $\ulcorner(S\supset S')\urcorner$ is a classical theorem) we may get an inferential hypothetical of the form $\ulcorner(TS\rightarrow TS')\urcorner$ in PA2. This entails, of course, that in certain cases, we also have theorems asserting something related to the old paradoxes of strict and material implication. For example, in PA2, "If (p.-p) is true, then q is true" holds, and the old dictum "If a contradiction were true, then anything whatever would be true," is precisely, and rigorously expressed and demonstrated; thus those who hold this principle can have no complaints.

But fifthly this does not mean that PA2 contains any paradoxes of strict and material implication as they occur in classical logic. For there is a difference between saying:

(1) "If both p and not p <u>were true</u> then q would be <u>true</u>"

and saying

(2) "If p and not p, then q would be true"

or

(3) "If it were true that (both p and not p,) then q"

or simply

(4) "If p and -p, then q"

which last is, strictly speaking, what has been called a paradox of strict implication. Although (1) holds, neither (2) nor (3) nor (4) are theorems in PA2. For PA2 rejects Tarski's conception of truth - that 'p' is true if and only if p; this theory springs from restrictions rooted in truth-functional logic. A proposition 'p' does not mean the same as the proposition ''p' is true'; "Once upon a time there was a wicked witch" does not mean the same as "'Once upon a time there was a wicked witch' is true", as any child will testify. If this is granted, it may be argued that the logical truth of (1) and the failure of (4) is a function of the logic of the operator "It is true that...". Or, more philosophically, we may say that the notion of truth is such that while the positing that a contradiction was <u>true</u> would permit an inference to the <u>truth</u> of anything whatsoever, it is not necessarily a logical truth that '(p.-p)' implies everything, (i.e., that '(p.-p)→q' is <u>logically</u> true). A contradiction, '(p.-p)', is one thing, a statement that a contradiction is true 'T(p.-p)' is another, which though false, is not a contradiction itself. It is the latter, not a contradiction in itself, that entails the truth of anything.

In any case, the so-called paradoxes of material and strict implication are rigorously excluded from PA2, though in a sophisticated and intricate manner:

a) Since PA2, like CC1 and PA1, is complete with respect to classical logic, it contains all classical truth-functional truths. But properly understood, i.e., understood as equivalent to expressions using only denial and conjunction, these contain no "paradoxes" at all.

b) When the new conditional of PA2 replaces the truth functional conditional in the so-called "paradoxes" of material and strict implication, the resulting formulae all fail in PA2, as they did in PA1 and CC1.[11]

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[11] C. I. Lewis, in "Interesting Theorems in Symbolic Logic", The Journal of Philosophy and Scientific Method, Vol. X, 1913, pp. 239-242, listed 35 of these so-called paradoxes. He included at least three additional statements in Lewis and Langford, Symbolic Logic, New York, 1932, pp. 142-45, i.e., theorems 15.31, 15.61, 15.72. In the same book, pp. 174-5, he lists, using a modal operator, four "paradoxes of strict implication". All 38 of the first set, and all correlates of the second set with necessary propositions replaced by tautologies of PA2, fail by the truth-tables of the matrices above, when the truth-functional, or strict conditionals are replaced by the conditional of PA2.

Thus, for example,

a) '$(-p \supset (p \supset q))$' is a theorem of PA2.
a') '$(-p \rightarrow (p \rightarrow q))$' is not a theorem of PA2.
b) '$((p.-p) \supset q)$' is a theorem of PA2.
b') '$((p.-p) \rightarrow q)$' is not a theorem of PA2.

c) Material "paradoxes" do not become theorems for the new conditional when truth operators are added to antecedent and consequent. Thus, for example, all of the listed paradoxes of material implication, when converted into forms like

a'') $(T-p \rightarrow T(p \rightarrow q))$

a''') $(T-p \supset T(p \rightarrow q))$

still fail, as can be shown by the truth-tables. Thus "It is true that p is false," does not imply "It is true that if p then q".

d) On the other hand, all of the so-called paradoxes do hold if they are phrased, as in the following example:

$$a'''')\quad (T{-}p \rightarrow T(p \supset q))$$

which says that if it is true that p is false, then it is true that not both p and not q. Inferences of this sort held in PA1, but without the truth operator. There was no statement in the object language related to this fact.

This sort of principle is closely related to the advantage, in PA2, that the principle of simplification finds expression as

$$\text{Axiom 8.}\quad T(T(p.q) \rightarrow Tq)$$

which asserts that if both p and q are true, then q is true.

A sixth consideration in favor of PA2, over CC1, is that it is possible to prove the non-derivability of those formulae which would allow the undesirable inferences from $(p \rightarrow q)$ and $(r \rightarrow q)$ to the falsity of $(p \rightarrow -r)$.[12]

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[12]This is established by the following matrices which satisfy the axioms of PA2 but do not satisfy the class of formulae in question.

-	p
6	1
5	2
4	3
3	4
2	5
1	6

T	p
3	1
3	2
3	3
5	4
5	5
5	6

p→q	1	2	3	4	5	6
1	3	5	5	5	5	5
2	5	3	5	5	5	5
3	5	5	3	5	5	5
4	3	3	5	3	5	5
5	3	5	3	5	3	5
6	5	3	3	5	5	3

p.q	1	2	3	4	5	6
1	1	2	1	6	5	6
2	2	2	2	6	6	6
3	1	2	3	4	5	6
4	6	6	4	4	6	6
5	5	6	5	6	5	6
6	6	6	6	6	6	6

A seventh consideration is that it would be possible, if desired, to add axioms consistent to PA2 until enough were present to prove that all tautologies in the first matrices given were theorems, and thus to prove Post-completeness for the resulting system. But at this time this is not desirable. For we know that some tautologies of the first matrix (e.g., the formula mentioned above, i.e., $(T((p \rightarrow q).(r \rightarrow q)) \rightarrow T{-}(p \rightarrow -r))$), although not derivable in PA2 (cf. above), would become derivable in such a system. Thus it is probably best to consider PA2 as a possibly incomplete system pending further investigation.

Finally, it is possible to include within the theorems of PA2 something which closely resembles the deduction theorem, namely,

$$(T(S_1.S_2...S_n)\rightarrow TR)\rightarrow(T(S_1.S_2....S_{n-1})\rightarrow T(S_n \supset R))$$

Exactly what this means, and how it is related to the usual deduction theorems remains to be determined. But it is assuredly a theorem in PA2.

PA2 cannot be rejected on the grounds that it excludes anything; for it includes all that classical logic or intuitionistic logic include. It is a richer system, containing theorems which are independent of and not reducible to any theorems in classical logic. The main question is whether the additional primitives and theorems are worth being added to the corpus of classical logic. This leads us to the final section of this paper.

§ 6. The Law of Conditional Non-Contradiction and Subjunctive Conditionals.

In conclusion I wish to show that there is a value in adding the Law of Conditional Non-Contradiction to the corpus of established logic because it would solve many of the difficulties raised by subjunctive conditionals. The difficulties I have in mind are those discussed by Chisholm, Goodman, Quine, and many others. The conditionals involved are usually called counter-factual, or contrafactual, conditionals because they need not, like the truth-functional conditional, be true when the antecedent is false. But this terminology comes from the relative contrast of subjunctive conditionals with truth-functional conditionals in relation to fact. As Goodman has pointed out, it is misleading terminology, for the same difficulties occur whether or not the antecedent is false.[13]

[13] Goodman, Nelson, Fact, Fiction and Forecast, Harvard, 1955, p. 14.

Since the conditionals involved are most often expressed in the subjunctive mood and since they have frequently been called subjunctive conditionals in the literature, I shall use this terminology. No metaphysical or non-linguistic implications are intended. What I wish to show is that when we consider the distinctive logical (vs factual) properties of subjunctive conditionals, they are seen to have an intimate relation to the Law of Conditional Non-Contradiction.

As we shall see, the distinguishing logical, as opposed to factual, mark of subjunctive conditionals is that two conditionals with the same antecedents and contradictory consequents are incompatible. This principle of conditional contrariety may be expressed as,

$$-((p \rightarrow q).(p \rightarrow -q))$$

i.e., "It is false that both if p then q and if p then not q", which is a theorem in PA2.

First, note the intimate connection of this principle with the Law of Conditional Non-Contradiction. We assume that subjunctive conditionals, like all other conditionals, must obey the rule of the hypothetical syllogism and at least the intuitionist rules of transposition. We also assume that a subjunctive conditional, like any other, is false if its antecedent is true while its consequent is false. Thus the following are assumed logically true for all subjunctive conditionals:

(1) $-(((p \rightarrow q).(q \rightarrow r)).-(p \rightarrow r))$

(2) $-((p \rightarrow -q).-(q \rightarrow -p))$

(3) $-((p \rightarrow q).(p.-q))$

The addition of the Law of Conditional Non-Contradiction:

(4) $-(p \rightarrow -p)$

now implies the Law of Conditional Contrariety:

$$-((p \rightarrow q).(p \rightarrow -q))$$

For, substituting in (1), we get

(5) $-(((p\rightarrow q).(q\rightarrow -p)).-(p\rightarrow -p))$

whence, by (5) and (4), and affirming a disjunct, we get

(6) $-((p\rightarrow q).(q\rightarrow -p))$

but by the equivalence of '$(p\rightarrow -q)$' and '$(q\rightarrow -p)$' this is equivalent to

(7) $-((p\rightarrow q).(p\rightarrow -q))$

which is the law of conditional contrariety. Not only is this law derivable from the Law of Conditional Non-Contradiction; the Law of Conditional Non-Contradiction follows from it, by the principle of identity. For, substituting 'p' for 'q' in (7) we get

(8) $-((p\rightarrow p).(p\rightarrow -p))$

from which, with

(9) $(p\rightarrow p)$

(4) follows by affirming the disjunct. Thus the two principles are connected by quite unexceptional logical laws and principles. We cannot have '$-(p\rightarrow -p)$' without '$-((p\rightarrow q).(p\rightarrow -q))$' unless we are willing to have a conditional for which either the rule of syllogism, the intuitionist rule of transposition or the law of identity fails.

Now the principle of conditional contrariety occupies a very central role in discussions of subjunctive, contrafactual and, generally, non-truth-functional conditionals. Nelson Goodman, in his definitive article on "The Problem of Counterfactual Conditionals" says,

> "...the problem is to define the circumstances under which a given counterfactual holds while the opposing conditional with the contradictory consequent fails to hold."[14]

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[14]Goodman, Opus cit., p. 14.

To be sure, Goodman assumes throughout the sequel that the laws of logic are exhausted by classical logic with the truth-functional conditional, so that the "circumstances" he seeks are non-logical relevant factual conditions and non-logical physical or causal laws. Nevertheless, what he is seeking is a criterion of the circumstances under which the law of conditional contrariety would hold as a factual or probable truth. And it is significant that, though he believes that no adequate philosophy of science can be formulated without a solution to this problem, he confesses in conclusion an inability to see any way to meet the difficulties which arise by this formulation of the problem. For as long as he assumes that the basic conditional must be the truth-functional conditional, he is unable to define satisfactorily a set of conditions which would not yield both 'if p then q' and 'if p then not q'.

B. J. Diggs' article on counterfactual conditionals implicitly supports a similar point with respect to generalized conditionals. He introduces the problem with the following example. Suppose no one has ever jumped out of a certain window, so that '-(Ex)Jx' is true. Now in classical logic '-(Ex)Jx' implies

(1) $(x)(Jx \supset Kx)$

where K is any predicate whatever. Hence both

(2) $(x)(Jx \supset Hx)$

i.e., "If anyone jumps out of the window, then he is hurt" and

(3) $(x)(Jx \supset -Hx)$

i.e., "If someone jumps out of the window, then he is not hurt", are true by virtue of the truth-functional conditional. But this, says Diggs, does not coincide with our normal intent in asserting "If anyone had jumped out of this window, he'd have been hurt" for, "Our normal intention in joining one particular consequent function to the antecedent function is frustrated. On this account, it certainly appears that the suggested form of the generalized conditional fails to express adequately the meaning of the original counterfactual."[15]

[15] B. J. Diggs "Counterfactual Conditionals", Mind, Vol. LXI, 1952, p. 513.

In other words, what normally we intend in such cases is that the law of conditional contrariety should obtain. Diggs, like Goodman, works at the problem within the framework of classical logic. He does not claim to solve it, but holds that the problem of counterfactuals reduces to that of specifying what we mean by non-analytic "lawlike" statements.

Reichenbach in "Nomological Statements and Admissible Operations"[16] like

[16] Reichenbach, Hans, Nomological Statements and Admissible Operations, North-Holland Publishing Company, 1954.

many others, attempts to deal with the latter problem, still on the assumption that classical logic exhausts all logical truths. He gives the name "nomological implication" to the class of implications which either express a logical entailment, or express a law of nature, and further distinguishes a sub-class of nomological statements, called "admissible statements" which he regards as supplying "reasonable" propositional operations. The latter include operations which use conditionals which are counterfactual. He then says,

"Since conversational language is rather clear and unambiguous in the usage of conditionals contrary to fact, we possess in this usage a sensitive test for the adequacy of the explication of reasonable implications, and we shall often make use of it. For instance, it is required for a conditional contrary to fact that it be unique. By this property I mean that, if the implication '$a \supset b$' is used for a conditional contrary to fact, the contrary implication '$a \supset \bar{b}$' cannot be so used. Obviously, adjunctive (truth-functional) implication does not satisfy the condition of uniqueness when it is used counterfactually, because if 'a' is false, both contrary implications are true in the adjunctive (truth-functional) sense. It has often been pointed out that this absence of uniqueness makes adjunctive implications inappropriate for counterfactual use. In the theory of admissible implications it will therefore be an important requirement that two contrary implications cannot both be admissible. The present theory satisfies this requirement, whereas my previous theory could only satisfy it to some extent."[17]

[17] Opus Cit., pp. 7-8.

Thus, the principle of conditional contrariety is a definitive element in Reichenbach's theory. However, the determination of whether a particular classical conditional, '$(p \supset q)$' statement is nomological depends upon prior empirical determinations with respect to various universal statements, and derivatively, whether or not it is false that both '$(p \supset q)$' and '$(p \supset -q)$' are true depends upon complicated decisions on matter of fact. Only after this, among other synthetic matters, has been decided can it be decided whether or not the implication is an "Admissible nomological implication".

Reichenbach developed his theory in the metalanguage, but considers in conclusion the possibility of an axiomatic formulation for "reasonable implication" with an arrow symbolizing this sort of "if...then..." with a restricted meaning. His definition of this sort of "if...then..." entailed the principle of conditional non-contrariety. But he gave up the attempt to construct such a calculus as not practicable since arrow-implications can only be applied after extensive empirical checking, and since his formulation leads to limitations with respect to transposition as well as failing for the principle of invariance.[18]

[18] Reichenbach, Hans, Nomological Statements and Admissible Operations, North-Holland Publishing Company, 1954.

"There exists a second way of introducing a reasonable implication, according to which an 'if-then' statement has a restricted meaning. ... As far as relative admissible implications are concerned, the empirical conditions referred to even include matters of fact, expressed in the high probability of the major antecedent, and in addition, the conditions of separability. We therefore can assign meaning to such an implication only when the empirical conditions are known to be satisfied. I will speak here of physical meaning.

An implication of this kind will be called proper implication. Since it will be denoted by an arrow, it may also be called arrow-implication. A sentence 'a→b' is an ambivalent expression; it may be true, false, or meaningless, depending on certain empirical conditions. The definition of proper implication is given as follows

Definition 48.

α) The statement 'a→b' is true if and only if 'a⊃b' is permissible.

β) The statement 'a→b' is false if and only if 'a→b̄' is true.

γ) In all other cases, 'a→b' is meaningless.

The arrow implication, or proper implication, is especially suited for the expression of a conditional contrary to fact." p. 120

"Furthermore, it was shown that contraposition is subject to serious limitations; we refer to theorem 17-18 and 23-24. And in the discussion of (80) and (81) it was shown that the invariance principle of implication does not hold for admissible implications.

This survey shows that the operations with proper implications are rather limited and require attention to the distinction between various kinds and orders, and to possible reducing. It is therefore not possible to construct a practicable calculus of arrow-implication. In general, if we want to construct derivations, we have to use the familiar adjunctive calculus; and we then must check for individual results whether they can be interpreted by arrow-implications. The class of statements which conversational language regards as reasonable is not complete; in order to construct derivative relations between statements of this class we have to go beyond the class." p. 123.

Although not all writers on subjunctive conditionals have explicitly mentioned the principle of conditional contrariety, it is an implicit premiss in most of the problems which are raised. One central problem was raised by Quine in this way:

> "It may be wondered, indeed, whether any really coherent theory of the contrafactual conditional of ordinary usage is possible at all, particularly when we imagine trying to adjudicate between such examples as these:
>
> If Bizet and Verdi had been compatriots, Bizet would have been Italian
>
> If Bizet and Verdi had been compatriots, Verdi would have been French. "[19]

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[19]Quine, W.V.O., Methods of Logic, New York, 1950, pp. 14-15.

Chisholm also gives a pair of conditionals involving a similar problem, saying that this type of case, so far as he could see, would "break any indicative formula we may devise, whatever its complexity":

> If Apollo were a man, he would be mortal.
> If Apollo were a man, at least one man would be immortal. [20]

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[20] Chisholm, R. ... p. 494

And additional pairs of statements are presented by Goodman,[21] and many others.

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[21] Opus Cit., pp. 15, 22.

Now quite apart from how the problem raised by these examples is to be solved, it must be pointed out that there would be no problem at all unless it were assumed that for this sort of conditional "If p then q" and "If p then not q" would be incompatible. For Quine's problem is only a problem as he intends it, if we assume that we must adjudicate between them, and we must adjudicate between them presumably because

(1) If Bizet and Verdi had been compatriots then Bizet would have been Italian

is held to imply

(2) If Bizet and Verdi had been compatriots, then Bizet would not have been French

and thus, (from the meaning of "compatriots"),

(3) If Bizet and Verdi had been compatriots, Verdi would not have been French

which is the conditional contrary of Quine's second statement, (or, because the second statement similarly implies the conditional contrary of the first). Careful analysis will show that the same principle of subjunctive contrariety must be considered implicit in Chisholm's and Goodman's presentations of this problem. Were we not considering subjunctive conditionals none of these pairs would present problems, for we do not claim that truth-functional conditionals are incompatible when they have identical antecedents and incompatible consequents. Thus this principle of conditional contrariety is apparently presupposed as a definitive and implicit logical characteristic of subjunctive conditionals.

There are therefore very good grounds for holding that logics which contain the Law of Conditional Non-Contradiction, and with it the principle of conditional contrariety, may plausibly be construed as logics which axiomatise subjunctive conditionals. And if subjunctive conditionals do play an important role in inference, there would be value in extending and revising classical logic to include such axiomatizations within formal logic. Efforts to handle subjunctive conditionals within the limits of classical logic are bound to end in frustration and failure. But if they are recognized as legitimate primitives in formal logic, as in PA2, the logical problems of subjunctive conditionals disappear. Epistemologically, there will be no difficulty in determining falsehood conditions. There will be epistemological problems in talking about truth-conditions, but these problems are already present anyway. The expressions of ordinary language and science will be directly and naturally expressible in the language of this logic. The requirement, sometimes suggested for natural laws, that natural laws should entail subjunctive conditionals is solved quite simply by using the subjunctive conditional in the matrix of universal generalizations and applying the rule of universal instantiation. As for the problem of Quine and others, the adjudication between contrary conditionals will be decidable by logic if and only if one or the other is analytically true or false. But synthetically Quine's problem is no more difficult to handle than the problem of how to decide between synthetic contraries in classical logic.

Suppose we have two classical contraries

(1) There are objects on Jupiter but none are alive.

(2) There are objects on Jupiter and some are alive.

These cannot both be true together because they are logical contraries. Must we adjudicate between them? Well, in fact we suspend judgment at present, but we agree they could be decided. And the way to decide them would be to add more facts than we now have about the objects on Jupiter. Quine's examples are analogous. As they stand, we cannot adjudicate between them, though they be contraries. We have to have more information. But the information we need is not found by looking for facts about Verdi and Bizet, but by finding out what other assumptions are intended to be held fixed in the context in which these conditionals are stated. If the additional fixed assumptions turn out to include

Bizet was French,

then we adjudicate in favor of the second conditional; if the assumptions intended to be held fixed include

Verdi was Italian,

then we decide in favor of the first. If the assumptions are self-contradictory we can't decide. When consistent missing premisses are inserted in the antecedent, the conditionals become logically adjudicable, just as, when the missing evidence and assumptions relating to (1) and (2) above are added, decision follows by logical considerations.

The principles of Conditional Non-Contradiction and conditional contrariety have been approached from two different sources. Nelson and McCall, both interested in the theory of analytic truth, or logical entailment, have both wished to develop a formal logic containing these principles. Their motives were to capture the notion of a connexion between antecedent and consequent, mentioned by Sextus Empiricus as one of several notions of implication in the 4th century B.C., such that "A implies B", if and only if A is incompatible with B.[22] On

[22]Sextus Empiricus, Hyp. Pyrrh. II, 110.

the other hand, as we have just pointed out, the principle of conditional contrariety has occupied a central role in the problem of handling synthetic contrary-to-fact or subjunctive conditionals, which are used in science. I hold that these two principles are neither exclusively connected with analytic entailments, nor are they exclusively connected with synthetic truths. Rather they are logical laws implicit in the subjunctive conditional as such. When a subjunctive conditional is found true by analysis of language alone, i.e., by logic, it is an entailment. When it is not logically true but is found true or false, or claimed to be true or false, or held to be probably true or false, on the basis of factual evidence, it is a synthetically true or false, or probably true or false conditional. Thus these conditionals, in my opinion, serve the purposes of both logic and science more fully and more precisely than any truth-functional conditionals, and formal logic should be extended to recognize them.